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Continuous dependence results for non-linear Neumann type boundary value problems

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Abstract

We obtain estimates on the continuous dependence on the coefficient for second-order non-linear degenerate Neumann type boundary value problems. Our results extend previous work of Cockburn et al., Jakobsen and Karlsen, and Gripenberg to problems with more general boundary conditions and domains. A new feature here is that we account for the dependence on the boundary conditions. As one application of our continuous dependence results, we derive for the first time the rate of convergence for the vanishing viscosity method for such problems. We also derive new explicit continuous dependence on the coefficients results for problems involving Bellman–Isaacs equations and certain quasilinear equation.

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1. Introduction

In this paper we will derive continuous dependence estimates for the following boundary value problem:

$$F\big(x, u, Du, D^2u\big) = 0 \quad \text{in } \Omega \quad \big(\Omega \subset \mathbb{R}^N\big), \tag{1.1}$$

$$G(x, Du) = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where u is the scalar unknown function, Du and D^2u denote its gradient and Hessian, and Ω is a bounded, smooth ($W^{3,\infty}$) domain in $\mathbb{R}^N$. Informally speaking, by continuous dependence estimates we mean estimates of the type

$$\|u_1 - u_2\| \leqslant \|F_1 - F_2\| + \|G_1 - G_2\|,$$

where u_1 and u_2 are solutions of two different boundary value problems with data F_1, G_1 and F_2, G_2. The exact statement is given in Section 2.

Eq. (1.1) is degenerate elliptic, (possibly) non-linear, and increasing in u. This means that the possibly non-linear function $F(x, r, p, X)$ satisfies

$$F(x, r, p, X) \leqslant F(x, s, p, Y) \quad \text{for all } r \leqslant s,\ X \geqslant Y,$$

where $x \in \Omega$, $r, s \in \mathbb{R}$, $p \in \mathbb{R}^N$, and $X, Y \in \mathbb{S}^N$. Here $\mathbb{S}^N$ is the set of real symmetric $N \times N$ matrices and $X \geqslant 0$ in $\mathbb{S}^N$ means that X is positive semi-definite. The boundary condition (1.2) satisfies the Neumann type condition that G is strictly increasing in p in the normal direction:

$$G\big(x, p + tn(x)\big) \geqslant G(x, p) + \nu t,$$

for some $\nu > 0$ and all $t \geqslant 0$, $x \in \partial\Omega$, p, and outward unit normal vectors $n(x)$ to $\partial\Omega$ at x. The other assumptions on F and G will be specified later.

The class of boundary conditions G we treat in this paper includes the classical Neumann condition, $\frac{\partial u}{\partial n} = g(x)$ in $\partial\Omega$, oblique derivative conditions, and non-linear boundary conditions like the capillary condition

$$\frac{\partial u}{\partial n} = \bar{\theta}(x)\big(1 + |Du|^2\big)^{1/2} \quad \text{in } \partial\Omega,$$

and the controlled reflection condition

$$\sup_{\alpha \in \mathcal{A}} \big\{\gamma_\alpha(x) \cdot Du - g_\alpha(x)\big\} = 0 \quad \text{in } \partial\Omega.$$

In this paper we will assume that $g, \bar{\theta}, g_\alpha, \gamma_\alpha$ are Lipschitz continuous functions, that $|\bar{\theta}| \leqslant \omega < 1$ and $\gamma_\alpha \cdot n \geqslant \nu > 0$, and that A is a compact metric space.

The main class of equations that our framework can handle are equations satisfying assumption (H2) in the next section. Loosely speaking, this is the class of equations where the non-linearity $F(x, r, p, X)$ is uniformly continuous in r, p, X locally uniformly in x. This case will be referred to as the "standard case" in the rest of this paper. Assumption (H2) excludes most quasilinear equations, but contains fully non-linear equations like the Bellman–Isaacs equations from optimal stochastic control and stochastic differential games theory:

$$\inf_{\theta_1\in\Theta_1}\sup_{\theta_2\in\Theta_2}\left\{-\operatorname{tr}\left[a^{\theta_1,\theta_2}(x)D^2u\right]-b^{\theta_1,\theta_2}(x)Du-c^{\theta_1,\theta_2}(x)u-f^{\theta_1,\theta_2}(x)\right\}=0\quad\text{in }\Omega,$$

where $\Theta_1,\Theta_2\subset\mathbb{R}^m$ are compact metric spaces, $c^{\theta_1,\theta_2}\geqslant\lambda>0$, the matrices $a^{\theta_1,\theta_2}\geqslant 0$, and the coefficients are Lipschitz continuous uniformly in θ_1,θ_2. In Sections 4 and 5, we give all the details, more examples, and extensions to problems on unbounded domains, time-dependent problems, and certain quasilinear equations like e.g.

$$-\operatorname{tr}\left[\left(I-\frac{Du\otimes Du}{1+|Du|^2}\right)D^2u\right]+\lambda u=f(x)\quad\text{in }\Omega.$$

Since these equations may be degenerate and non-linear, their solutions will in general not be smooth. In this paper we work with a concept of weak solutions called viscosity solutions, a precise definition is given at the end of this introduction. Viscosity solutions are at least continuous in the interior of Ω. The boundary conditions will be interpreted in the weak viscosity sense which essentially means that either the boundary condition or the equation has to hold on the boundary. This allow us to have well-posed problems even when the boundary conditions are classically incompatible. The solutions can realized by the vanishing viscosity method, and they will be discontinuous at parts of the boundary where the boundary conditions are classically incompatible.

An overview of the viscosity solution theory, including Neumann boundary value problems, can be found in the User's guide [12]. The viscosity solution theory for Neumann type boundary value problems was initiated by Lions [28] in 1985 for first-order equations, and has been developed by many authors since, see [1–3,15,17,19,29] and references therein for various aspects of this theory. Today there are two leading approaches, one due to Ishii [17] and another one due to Barles [1,2]. They apply under slightly different assumptions and will be discussed below.

Starting with the standard case, i.e. non-linear equations (1.1) satisfying (H2), we prove under natural and standard assumptions, that these boundary value problems have unique Hölder continuous viscosity solutions. The Hölder regularity results are new and extend the Lipschitz regularity result of Barles [1], and we give for the first time a complete proof of such a regularity result. We note that these regularity results are global up to the boundary. Local up to the boundary Hölder estimates have previously been obtained by Barles and Da Lio [3] for a different class of equations. Whereas our equations are degenerate but strictly increasing in the u argument (assumption (H3) in the next section), their equations are weakly non-degenerate satisfying some sort of "strong ellipticity condition" but are not necessarily increasing in u. The arguments needed to prove the two types of results are also different, except for some ideas on the construction of test functions that are needed in some of the proofs.

Next we prove continuous dependence results comparing Hölder continuous (sub- and super-) solutions of different boundary value problems. The results we obtain include both continuous dependence on non-linearities for the equation and the boundary condition. The results concerning the dependence on the boundary condition are completely new, at least in a viscosity solution setting, while the results we obtain for the equations apply to much more general boundary conditions (including non-linear ones) than earlier results.

Continuous dependence results for the type of equations we consider in this paper have previously been obtained by e.g. Cockburn et al. [11], Jakobsen and Karlsen [20–22], and Gripenberg [16]. In all these papers viscosity solution methods are used. In some cases such results can also be obtained from probabilistic arguments, see e.g. [14] for results for Bellman equations set in $\mathbb{R}^N$. Papers [20–22] treat very general classes of equations set in $\mathbb{R}^N$ or $\mathbb{R}^N \times [0, T)$, [11] treats zero-Neumann boundary value problem for x-independent equations, and [16] treats a zero-Dirichlet boundary value problem.

In the two last papers the domain Ω is convex and possibly unbounded and in the last paper further restrictions on the class of equations are needed (because of the Dirichlet condition) and the Dirichlet condition is taken in the classical sense. All these papers treat more general quasilinear equations than we can treat here, e.g. p-Laplace type equations for $p > 2$.

The technical explanation for the differences between our continuous dependence result and the above mentioned results lays in the choice of test function we use. To handle weakly posed Neumann boundary conditions, the idea is to use a test function that will never satisfy the boundary condition. The effect in the proofs will be that the equation holds also at the boundary, and that the classical viscosity solution comparison argument can be used (see the following sections). To achieve this the usual test function has to be modified and the extent of the modifications depend on how smooth and non-linear the Neumann condition is. To handle possibly non-linear boundary conditions or Hölder continuous solutions in combination with boundary reflection directions that are only Lipschitz functions in the space variable, it seemed that the only available or at least the most optimal test function to use, is the one constructed by Barles in [1,2]. As opposed to the basic test function used in the other papers on continuous dependence, the test function of Barles is not symmetric in its arguments (x and y) and therefore it does not have equal x and y gradients. We loose a cancellation property in the comparison proof and hence cannot handle as general gradient dependence in the equations as with the basic test function. In this paper we consider the same class of non-singular(!) equations as Barles in [1,2], and this excludes most of the quasilinear equations considered in [11,16,20,21], including p-Laplace equations for $p \neq 2$ (see also Remark 5.1 in Section 5).

At this point we mentioned that a different test function has been constructed by Ishii in [17]. Compared with Barles, Ishii is able to treat less regular domains but with more regular (and less non-linear) boundary conditions (e.g. C^1 domains and $W^{2,\infty}$ reflections), see [1] for a more detailed comparison. Using Ishii's test function, continuous dependence results could probably be obtained under a different set of assumptions (see above). We have not considered this case.

We also point out that we can handle u-depending boundary conditions only through additional arguments involving transformations. This is in contrast to the general *comparison* results obtained by Barles [2] under similar assumptions for F and G. In [2] u-dependence is handled directly by a sort of localization argument [2, Lemma 5.2] which only works when you send some parameter of the test function to zero. In our continuous dependence arguments, we will have to optimize with respect to this parameter and the optimal choice will in general not be zero or even small. See the treatment of parameter ε at end of the proof of Theorem 2.2. One way to handle u-depending Neumann type boundary value problems, is to transform them into problems with no u-dependence, then to use our results, and finally to transform back.

We do not consider such transformations in this paper, instead we refer to [3] where such transformations have been considered in a rather general setting.

Continuous dependence results have to do with well-posedness of the equation. Typically the boundary value problem we consider model some physical process, and the data is measured data. A continuous dependence result then implies that small measurement errors only produce small errors in the solutions. Any reasonable model should satisfy such a requirement in particular in view of numerical computations. Moreover, continuous dependence results have been used in many other contexts. They play a key part in the shaking of coefficients approach of Krylov to obtain error estimates for approximation schemes for Bellman equations [4–6,24–26], in Bourgoing [8] and in [20] they are used to obtain regularity results, and they have been used to estimate diffusion matrix projection errors [7], source term splitting errors [23], and errors coming from the truncation of Levy measures [22]. They have also been used to derive the rate of convergence for the vanishing viscosity method [11,16,20,21], see also e.g. [10].

The paper is organized as follows: In the next section we state the assumptions on the boundary value problem (1.1) and (1.2) in the standard case and give well-posedness and Hölder regularity results. We state the main result, the continuous dependence result, and as an immediate corollary we derive an estimate on the rate of convergence for the vanishing viscosity method. The proofs of the main result along with the regularity result are proven in Section 3, and in Section 4 we apply our main result to obtain new continuous dependence results for boundary value problems involving Bellman–Isaacs equations. We give several extensions of our results in Section 5, to time-depending equations, equations set on unbounded domains, and certain quasilinear equations. Finally, in Appendix A we derive the test function used in the proofs in Section 3 along with its properties.

Notation

We let $|\cdot|$ denote the Euclidean norm both in $\mathbb{R}^m$ (vectors) and $\mathbb{R}^{m\times p}$ (matrices) for $m, p \in \mathbb{N}$. We denote by $\mathbb{S}^N$ the space of symmetric $N\times N$ matrices, tr and ^{T} denote trace and transpose of matrices, and $\leqslant$ denote the natural orderings of both numbers and square matrices. For $a, b \in \mathbb{R}$ we define $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. We will also denote various constants by K or C, and their values may change from line to line.

Let $BUSC(U)$, $BLSC(U)$, $C(U)$, and $W^{p,\infty}(U)$ denote the spaces of bounded upper- and lower-semicontinuous functions, continuous functions, and functions with p essentially bounded derivatives, all functions defined on U. If $f : \mathbb{R}^N \to \mathbb{R}^{m\times p}$ is a function and $\alpha \in (0, 1]$, then define the following (semi) norms:

$$|f|_0 = \sup_{x\in\bar{\Omega}} \bigl|f(x)\bigr|, \quad [f]_\alpha = \sup_{\substack{x,y\in\bar{\Omega}\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}, \quad \text{and} \quad |f|_\alpha = |f|_0 + [f]_\alpha.$$

By $C^{0,\alpha}(\bar{\Omega})$ we denote the set of functions $f : \bar{\Omega} \to \mathbb{R}$ with finite norm $|f|_\alpha$.

We end this section by recalling the definition of a viscosity solution:

Definition 1.1. An upper-semicontinuous function u is a *viscosity subsolution* of (1.1) and (1.2) if for all $\phi \in C^2(\bar{\Omega})$, at each maximum point $x_0 \in \bar{\Omega}$ of $u - \phi$,

$$F\bigl(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)\bigr) \leqslant 0 \quad \text{if } x_0 \in \Omega, \tag{1.3}$$

$$\min\bigl(F\bigl(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)\bigr), G\bigl(x_0, Du(x_0)\bigr)\bigr) \leqslant 0 \quad \text{if } x_0 \in \partial\Omega. \tag{1.4}$$

A lower-semicontinuous function u is a *viscosity supersolution* of (1.1) and (1.2) if for all $\phi \in C^2(\bar{\Omega})$, at each minimum point $x_0 \in \bar{\Omega}$ of $u - \phi$,

$$F\big(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)\big) \geqslant 0 \quad \text{if } x_0 \in \Omega, \tag{1.5}$$

$$\max\Big(F\big(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)\big), G\big(x_0, Du(x_0)\big)\Big) \geqslant 0 \quad \text{if } x_0 \in \partial\Omega. \tag{1.6}$$

Finally u is a solution when it is both a super- and a subsolution.

2. The main results

In this section we consider the standard case (when assumption (H2) below holds). Following [1,2] we state the assumptions on the boundary value problem (1.1) and (1.2) and give results on comparison, uniqueness, and existence of solutions. Then we give new Hölder regularity results extending the Lipschitz regularity result of [1] in two ways: we allow Hölder continuous data and small λ (see assumption (H3) below). We also give a complete proof. The main result of this paper, the continuous dependence result, is then stated, and as an immediate consequence we derive an explicit rate for the convergence of the vanishing viscosity method.

Here is a list of the assumptions we will use, starting by the domain:

(H0) Ω is a bounded domain in $\mathbb{R}^N$ with a $W^{3,\infty}$ boundary.

For the equation we use the following standard assumptions:

(H1) $F \in C(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N)$.

(H2) There exists a modulus $\omega_{R,K}$ (a continuous, non-decreasing function satisfying $\omega_{R,K}(0) = 0$) such that

$$F(y, r, q, Y) - F(x, r, p, X) \leqslant \omega_{R,K}\left(|x - y| + \frac{1}{\varepsilon^2}|x - y|^2 + \eta^2 + \varepsilon^2 + B\right),$$

for $\varepsilon, \eta \in (0, 1]$, $B \geqslant 0$, $x, y \in \bar{\Omega}$, $r \in \mathbb{R}$, $|r| \leqslant R$, $p, q \in \mathbb{R}^N$ and $X, Y \in \mathbb{S}^N$ satisfying $|x - y| \leqslant K\eta\varepsilon$, $|p - q| \leqslant K(\eta^2 + \varepsilon^2 + B)$, $|p| + |q| \leqslant K(\frac{\eta}{\varepsilon} + \eta^2 + \varepsilon^2 + B)$, and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant \frac{K}{\varepsilon^2}\begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{pmatrix} + K\big(\eta^2 + \varepsilon^2 + B\big)\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}. \tag{2.1}$$

($\overline{\text{H2}}$) There exist $\alpha \in (0, 1]$ and $K_R \geqslant 0$ such that

$$F(y, r, q, Y) - F(x, r, p, X) \leqslant K_R\left(|x - y|^\alpha + \frac{1}{\varepsilon^2}|x - y|^2 + \eta^2 + \varepsilon^2 + B\right),$$

where $\varepsilon, \eta, B, R, x, y, p, q, X, Y$ are as in (H2).

(H3) For every x, p, X, and for any $R > 0$, there is $\lambda_R > 0$ such that

$$F(x, r, p, X) - F(x, s, p, X) \geqslant \lambda_R(r - s) \quad \text{for } -R \leqslant s \leqslant r \leqslant R.$$

The possibly fully non-linear Neumann type boundary condition satisfies:

(HB1) There exists $\nu > 0$ such that for all $\mu > 0$, $x \in \partial\Omega$, $p \in \mathbb{R}^N$,

$$G\big(x, p + \mu n(x)\big) - G(x, p) \geqslant \nu\mu,$$

where $n(x)$ is the unit outward normal at x.

(HB2) There exists a constant K such that for all $x, y \in \partial\Omega$ and all $p, q \in \mathbb{R}^N$,

$$\big|G(x, p) - G(y, q)\big| \leqslant K\big[\big(1 + |p| + |q|\big)|x - y| + |p - q|\big].$$

Remark 2.1. In general there is a trade off between the regularity of the boundary $\partial\Omega$ and the generality and smoothness of the boundary condition G, see [1] for a discussion. (H0) compensates for very general non-smooth boundary conditions.

Remark 2.2. Assumption (H2) plays the same role as (3.14) in the User's guide [12]. By this assumption the equation is degenerate elliptic. Moreover, it is a refined version of assumption (H5-1) in [2] containing also a new parameter B. In the proofs, this parameter will be used to carry information from the boundary conditions (which are never satisfied in the proofs, see the introduction) over to the equations. Assumption ($\overline{\text{H2}}$) is a strengthening of hypothesis (H2) which yields Hölder regularity results.

By (H3) the equation is strictly increasing in the u argument. Assumption (HB1) is the Neumann assumption, saying that the boundary condition G contains non-vanishing and non-tangential (to $\partial\Omega$) oblique derivatives and it is a natural condition to insure the well-posedness of the problem.

We now state a comparison, uniqueness, existence, and regularity result for solutions of (1.1) and (1.2).

Theorem 2.1. *If* (H0)–(H3), (HB1), *and* (HB2) *hold, then the following statements are true*:

(a) *If u is a $BUSC(\bar{\Omega})$ subsolution and v is a $BLSC(\bar{\Omega})$ supersolution of* (1.1) *and* (1.2), *then $u \leqslant v$ in $\bar{\Omega}$.*
(b) *If λ_R in* (H3) *is independent of R, then there exists a unique solution $u \in C(\bar{\Omega})$ of* (1.1) *and* (1.2).
(c) *Assume* ($\overline{\text{H2}}$) *also holds, $u \in C(\bar{\Omega})$ is the solution of* (1.1) *and* (1.2), *and $\lambda := \lambda_{|u|_0} > 0$. Then there are constants $\beta \in (0, \alpha]$ and K (only depending on the data and λ) such that*

$$\big|u(x) - u(y)\big| \leqslant K|x - y|^{\beta} \quad \text{in } \bar{\Omega} \times \bar{\Omega}.$$

Furthermore, there exists a constant $\bar{\lambda} > 0$ (only depending on the data) such that if $\lambda > \bar{\lambda}$ then $\beta = \alpha$ (the maximal regularity is attained).

The comparison principle in (a) correspond to Theorem 2.1 in [2]. The uniqueness part in (b) follow from (a), and existence follows from Perrons method [18] since $w(x) := M - Kd(x)$ is a supersolution of (1.1) and $-w$ is a subsolution of (1.2), if $M, K \geqslant 0$ are big enough, and d is the $W^{3,\infty}$ extension of the distance function defined in Appendix A, see Section 4 in [2] for similar results. The regularity result, part (c), will be proved in Section 3.

Remark 2.3. The regularity results in part (c) are global up to the boundary. Local up to the boundary Hölder estimates have been obtained by Barles and Da Lio [3] using different techniques and assumptions on the non-linearity of the equation. See the introduction for a discussion.

Now we proceed to the continuous dependence result. We will derive an upper bound on the difference between a viscosity subsolution u_1 of

$$\begin{aligned} F_1\big(x, u_1(x), Du_1(x), D^2u_1(x)\big) &= 0 \quad \text{in } \Omega, \\ G_1\big(x, Du_1(x)\big) &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.2}$$

and a viscosity supersolution u_2 of

$$\begin{aligned} F_2\big(x, u_2(x), Du_2(x), D^2u_2(x)\big) &= 0 \quad \text{in } \Omega, \\ G_2\big(x, Du_2(x)\big) &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.3}$$

We assume the following estimates on the differences of the two equations and of the two boundary conditions.

(D1) There are $\delta_1, \delta_2 \geqslant 0$, and $K_F(K) \geqslant 0$ such that for any $K \geqslant 0$,

$$F_2(y, r, q, Y) - F_1(x, r, p, X) \leqslant K_F(K)\left(\eta^2 + \delta_1 + \frac{1}{\varepsilon^2}\delta_2^2 + B\right),$$

for $0 < \varepsilon \leqslant \eta := \varepsilon^{\frac{\bar\alpha}{2-\bar\alpha}} \leqslant 1$ with $\bar\alpha = \alpha \wedge \beta$, $B \geqslant 0$, $x, y \in \bar\Omega$, $r \in \mathbb{R}$, $|r| \leqslant K$, $p, q \in \mathbb{R}^N$ and $X, Y \in \mathbb{S}^N$ satisfying $|x - y| \leqslant K\eta\varepsilon$, $|p - q| \leqslant K\eta^2 + KB$, $|p| + |q| \leqslant K(\frac{\eta}{\varepsilon} + \eta^2 + B)$, and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant \frac{K}{\varepsilon^2}\begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{pmatrix} + K\big(\eta^2 + B\big)\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}.$$

(D2) There are $\mu_1, \mu_2, K_G \geqslant 0$ such that for all $x \in \partial\Omega$ and $p \in \mathbb{R}^N$,

$$G_2(x, p) - G_1(x, p) \leqslant K_G\big(\mu_1 + \mu_2|p|\big).$$

Remark 2.4. Assumption (D1) is a "continuous dependence" version of (H2) and ($\overline{\text{H2}}$) in this paper, and assumption (3.14) in the User's guide [12]. A similar assumption is used in Theorem 2.1 in [21].

By β and α we denote the Hölder exponents of the solutions and data, respectively. In general $\alpha \geqslant \beta$, and equality only holds when λ in (H3) is big enough.

Since $|x - y| \leqslant K\varepsilon\eta$, $\eta = \varepsilon^{\frac{\beta}{2-\beta}}$ imply $\frac{|x-y|^2}{\varepsilon^2} \leqslant K\eta^2$ and $|x - y|^\beta \leqslant K\eta^2$, the F_1–F_2 inequality in (D1) will be implied by the following more standard inequality

$$F_2(y, r, q, Y) - F_1(x, r, p, X) \leqslant K\left(|x - y|^\alpha + \frac{1}{\varepsilon^2}|x - y|^2 + \delta_1 + \frac{1}{\varepsilon^2}\delta_2^2 + \eta^2 + B\right).$$

Remark 2.5. (On assumption (D2)). In the case of oblique derivative boundary conditions, $G_i(x,p) = \gamma_i(x)\cdot p - g_i(x)$, $i = 1,2$, and

$$\big|G_2(x,p) - G_1(x,p)\big| \leqslant \big|(g_1 - g_2)^+\big|_0 + |\gamma_1 - \gamma_2|_0|p|.$$

Our main result is stated in the following theorem:

Theorem 2.2 *(Continuous dependence estimate). Assume* (H0), (H1), (H3), (HB1), *and* (HB2) *hold for H_1, H_2, G_1, G_2, and $u_1, u_2 \in C^{0,\beta}(\bar{\Omega})$ for $\beta \in (0,1]$. Define $\nu^2 = (\nu_1 \vee \nu_2)(\nu_1 \wedge \nu_2)$ and $\lambda = \lambda_{1,|u_1|_0} \vee \lambda_{2,|u_2|_0}$.*

If (D1) *and* (D2) *hold and u_1 and u_2 satisfy the boundary value problems* (2.2) *and* (2.3) *respectively, then there exists a constant $C > 0$* (*depending only on K_F, K, K_G, $|u_1|_\beta$, $|u_2|_\beta$, α, β*) *such that*

$$\lambda \max_{\bar{\Omega}}(u_1 - u_2) \leqslant C\left(\delta_1 + \delta_2^{\alpha\wedge\beta} + \frac{\mu_1}{\nu} + \left(\frac{\mu_2}{\nu}\right)^{\alpha\wedge\beta}\right).$$

Remark 2.6. As far as we know this is the first result giving continuous dependence on the boundary condition. The result also extends the earlier continuous dependence on the equation type of results of [11,16,20,21] since much more general boundary conditions are considered (but at the expense of less general equations!).

We prove Theorem 2.2 in Section 3. An immediate consequence of this result is an estimate on the rate of convergence for the vanishing viscosity method. For $\mu > 0$ we consider the solution u_μ of

$$F\big(x, u, Du, D^2u\big) = \mu\Delta u \quad \text{in } \Omega, \tag{2.4}$$

with boundary condition (1.2). The result is the following:

Theorem 2.3. *Assume* (H0), (H1), ($\overline{\text{H2}}$), (H3), (HB1), (HB2), *$\mu > 0$, and that u and u_μ solve* (1.1)/(1.2) *and* (2.4)/(1.2), *respectively. Then u and u_μ belong to $C^{0,\beta}(\bar{\Omega})$ for some $\beta \in (0,\alpha]$ and*

$$|u - u_\mu|_0 \leqslant C\mu^{\beta/2}.$$

Proof. Regularity follows from Theorem 2.1. By assumption ($\overline{\text{H2}}$)

$$\big[F(y,r,q,Y) - \mu\operatorname{tr} Y\big] - F(x,r,p,X) \leqslant C\left(|x-y|^\alpha + \frac{|x-y|}{\varepsilon^2} + \eta^2 + \varepsilon^2\right) - \mu\operatorname{tr} Y,$$

and inequality (2.1) implies that $-\operatorname{tr} Y \leqslant C\frac{1}{\varepsilon^2} +$ small terms. Theorem 2.2 immediately gives $u - u_\mu \leqslant C\mu^{\beta/2}$. A lower bound can be found in a similar way. □

Remark 2.7. This result seems to be the first such result for complicated boundary conditions. We refer to [11,16] for results on weak 0-Neumann or classical 0-Dirichlet problems, to [30] for results on linear Neumann boundary value problems for first-order equations, and to [20,21] for result in $\mathbb{R}^N$ or $(0,T)\times\mathbb{R}^N$.

Remark 2.8. The vanishing viscosity method has been studied by many authors dealing with weak solutions of non-linear PDEs. The method has been used to obtain existence (and uniqueness!) of solutions for degenerate (e.g. first-order) problems by taking the limit as $\mu \to 0$ (see e.g. [9,31]), and it is well known that it is strongly related to the problem of proving convergence rates for numerical approximations of such problems (see e.g. [13,30]).

3. Proofs of Theorems 2.2 and 2.1(c)

Proof of Theorem 2.2. First we assume without loss of generality that

$$\delta_1, \delta_2, \frac{\mu_1}{\nu}, \frac{\mu_2}{\nu} \leqslant 1.$$

If this is not the case then the theorem holds since

$$u_1 - u_2 \leqslant \big(|u_1|_0 + |u_2|_0\big)\left(\delta_1 + \delta_2^{\bar{\alpha}} + \frac{\mu_1}{\nu} + \left(\frac{\mu_2}{\nu}\right)^{\bar{\alpha}}\right),$$

where $\bar{\alpha} = \alpha \wedge \beta$. Then we double the variables and consider

$$\psi(x, y) = u_1(x) - u_2(y) - \phi(x, y) \quad \text{and} \quad M = \max_{x,y\in\bar{\Omega}} \psi(x, y) = \psi(\bar{x}, \bar{y}),$$

where for $A, B \geqslant 0$,

$$\begin{aligned}\phi(x, y) = {} & \frac{1}{\varepsilon^2}|x-y|^2 + \frac{A}{\varepsilon^2}\big(d(x) - d(y)\big)^2 - B\big(d(x) + d(y)\big) \\ & - \tilde{C}_2\left(\frac{x+y}{2}, \frac{2(x-y)}{\varepsilon^2}\right)\big(d(x) - d(y)\big),\end{aligned}$$

and $\tilde{C}_2(x, p) = C_{2,a}(x, p)$ with $a = \eta\varepsilon = \varepsilon^{\frac{2}{2-\bar{\alpha}}}$ ($\eta = \varepsilon^{\frac{\bar{\alpha}}{2-\bar{\alpha}}}$ by (D1)). The functions $C_{2,a}$ and d are defined in Appendix A, and the smooth function ϕ was introduced by Barles in [2]. We refer to Appendix A for the proofs of the properties of ϕ.

The existence of a point $(\bar{x}, \bar{y})$ follows from compactness of $\bar{\Omega}$ and the continuity of all functions involved. Since $(\bar{x}, \bar{y})$ is a maximum point,

$$2\psi(\bar{x}, \bar{y}) \geqslant \psi(\bar{x}, \bar{x}) + \psi(\bar{y}, \bar{y}).$$

Moreover, if A is big enough, Lemma A.3 of Appendix A implies that

$$\phi(\bar{x}, \bar{y}) \geqslant \frac{1}{2\varepsilon^2}|\bar{x} - \bar{y}|^2 - K_0\varepsilon^2 - B\big(d(\bar{x}) + d(\bar{y})\big), \tag{3.1}$$

and Hölder regularity of u_1 and u_2 combined with the last two inequalities yield

$$\frac{1}{2\varepsilon^2}|\bar{x} - \bar{y}|^2 \leqslant K_1|\bar{x} - \bar{y}|^{\bar{\alpha}} \vee \varepsilon^2$$

for some constant K_1 depending on K_0 and the Hölder constants of u_1 and u_2 (but not on B). Equivalently, since $\eta = \varepsilon^{\frac{\tilde\alpha}{2-\tilde\alpha}}$ by (D1) and $\varepsilon \leqslant \eta$,

$$|\bar x - \bar y| \leqslant \tilde K_1 \varepsilon^{\frac{2}{2-\tilde\alpha}} = \tilde K_1 \eta\varepsilon \quad \text{and} \quad \frac{1}{\varepsilon^2}|\bar x - \bar y|^2 \leqslant \tilde K_1 \varepsilon^{\frac{2\tilde\alpha}{2-\tilde\alpha}} = \tilde K_1 \eta^2. \tag{3.2}$$

Now we choose A and B in the test function ϕ to insure that when $\bar x$ or $\bar y$ belong to the boundary $\partial\Omega$, then the boundary conditions cannot hold there. See Lemma A.4 of Appendix A. This means that the *equations* always has to hold at $\bar x$ and $\bar y$. The precise choices of A and B are

$$B = K\big(\eta^2 + \varepsilon^2\big) + \frac{K}{\nu}\left(\mu_1 + \mu_2\frac{\eta}{\varepsilon}\right) \quad \text{and} \quad A = K,$$

for some K only depending on the data of the problem.

By the maximum principle for semicontinuous functions, Theorem 3.2 of the User's guide [12], there are $(p, X) \in \bar J^{2,+}_{\bar\Omega} u_1(\bar x)$ and $(q, Y) \in \bar J^{2,-}_{\bar\Omega} u_2(\bar y)$ such that

$$p = D_x\phi(\bar x, \bar y), \qquad q = -D_y\phi(\bar x, \bar y),$$
$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant \big[\mathrm{Id} + \varepsilon^2 D^2\phi(\bar x, \bar y)\big] D^2\phi(\bar x, \bar y).$$

Using the definition of viscosity sub- and supersolutions at $\bar x$ and $\bar y$ (and Lemma A.4) we get

$$F_1\big(\bar x, u_1(\bar x), p, X\big) \leqslant 0 \leqslant F_2\big(\bar y, u_2(\bar y), q, Y\big).$$

We rewrite this as

$$F_1\big(\bar x, u_1(\bar x), p, X\big) - F_1\big(\bar x, u_2(\bar y), p, X\big) \leqslant F_2\big(\bar y, u_2(\bar y), q, Y\big) - F_1\big(\bar x, u_2(\bar y), p, X\big). \tag{3.3}$$

By Lemma A.5, the definitions of p, q, X, Y, and $\varepsilon \leqslant \eta \leqslant 1$, it follows that

$$|p - q| \leqslant K\eta^2 + 2B,$$
$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant \frac{K}{\varepsilon^2}\begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{pmatrix} + K\big(\eta^2 + B\big)\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix},$$

again for some K only depending on the data of the problem. Since we also have (3.2), we are in a position to use assumption (D1). So if $u_1(\bar x) - u_2(\bar y) \geqslant 0$, then (D1) and (H3) applied to (3.3) yield

$$\lambda_1\big(u_1(\bar x) - u_2(\bar y)\big) \leqslant K_F(K)\left(\eta^2 + \delta_1 + \frac{1}{\varepsilon^2}\delta_2^2 + B\right).$$

By (3.1) and the definition of ψ, it follows that

$$u_1(x) - u_2(x) \leqslant \psi_\varepsilon(x, x) \leqslant \psi_\varepsilon(\bar x, \bar y) \leqslant u_1(\bar x) - u_2(\bar y) + 2B\big(d(\bar x) + d(\bar y)\big).$$

Therefore the two previous inequalities and the choice of B implies that

$$\lambda_1\big(u_1(x)-u_2(x)\big)\leqslant K\left(\eta^2+\delta_1+\frac{1}{\varepsilon^2}\delta_2^2+\frac{[\mu_1+\mu_2\frac{\eta}{\varepsilon}]}{\nu}\right).$$

Remember that $\eta=\eta(\varepsilon)=\varepsilon^{\frac{\bar\alpha}{2-\bar\alpha}}$ and let ε_1 and ε_2 be defined by

$$\eta(\varepsilon_1)^2=\frac{1}{\varepsilon_1^2}\delta_2^2 \quad\text{or}\quad \eta(\varepsilon_1)^2=\delta_2^{\bar\alpha},$$
$$\eta(\varepsilon_2)^2=\frac{\mu_2\frac{\eta(\varepsilon_2)}{\varepsilon_2}}{\nu} \quad\text{or}\quad \eta(\varepsilon_2)^2=\frac{\mu_2^{\bar\alpha}}{\nu^{\bar\alpha}}.$$

Now with $\varepsilon=\varepsilon_1\vee\varepsilon_2$ ($\leqslant 1$ by assumption) it follows that

$$\lambda_1\big(u_1(x)-u_2(x)\big)\leqslant K\left(\delta_1+\delta_2^{\bar\alpha}+\frac{\mu_1}{\nu}+\frac{\mu_2^{\bar\alpha}}{\nu^{\bar\alpha}}\right).$$

A closer look at the proof reveals that we may replace λ_1 by $\lambda_1\vee\lambda_2$. $\square$

Proof of Theorem 2.1(c). We start by proving α-Hölder regularity when λ is big (the last statement of Theorem 2.1(c)). The proof is similar to the proof of Theorem 2.2 except that we have to modify the test function and use a bootstrap argument. The modified test function is

$$\begin{aligned}\phi_a(x,y)&=\frac{1}{\varepsilon^2}e^{-K_e(d(x)+d(y))}|x-y|^2+\frac{A}{\varepsilon^2}\big(d(x)-d(y)\big)^2\\&\quad-C_a\left(\frac{x+y}{2},\frac{2e^{-K_e(d(x)+d(y))}(x-y)}{\varepsilon^2}\right)\big(d(x)-d(y)\big)\\&\quad-K_B\big(a+\varepsilon^{\frac{2\alpha}{2-\alpha}}\big)\big(d(x)+d(y)\big).\end{aligned}$$

We refer to Appendix A for the definitions of C_a and d. Playing with the parameter a, we will use a bootstrap argument to prove that u has the right regularity.

The new test function satisfies similar estimates as the ones given in Lemmas A.3–A.5. The moral is that the new terms coming from the exponential term are not worse than the old terms. We refer to [2] for such estimates given in the full generality (but with a different choice of a).

Now let $\varepsilon\leqslant 1$ and double the variables defining

$$M:=\psi(\bar x,\bar y)=\sup\psi(x,y)\quad\text{where } \psi(x,y)=u(x)-u(y)-\phi_a(x,y).$$

If A is big enough, (an easy extension of) Lemma A.3 and the inequality $2\psi(\bar x,\bar y)\geqslant\psi(\bar x,\bar x)+\psi(\bar y,\bar y)$ imply that

$$\frac{1}{2\varepsilon^2}e^{-K_e(d(x)+d(y))}|\bar x-\bar y|^2\leqslant 2\big[u(\bar x)-u(\bar y)\big]+K_0\varepsilon^2\leqslant 2|u|_0+K_0. \tag{3.4}$$

Define $\eta^2 = K^{-1}\frac{1}{\varepsilon^2}|\bar{x}-\bar{y}|^2$ with $K = e^{2K_e}(2|u|_0 + K_0)$. By (3.4),

$$\eta^2 \leqslant 1 \quad \text{and} \quad |\bar{x}-\bar{y}| \leqslant K^{1/2}\eta\varepsilon.$$

We proceed as in the proof of Theorem 2.2.

By arguments similar to the ones in the proof of Lemma A.4, if A, K_e and K_B are big enough (not depending on ε, a or B), then the equation holds even if $(\bar{x},\bar{y})$ lies on $\partial(\Omega\times\Omega)$. Compared with the proof of Theorem 2.2, the exponential allows us to cancel at the boundary all terms of the form $\frac{1}{\varepsilon^2}|\bar{x}-\bar{y}|^2 = K\eta^2$ and use $B = K_B(a + \varepsilon^{\frac{2\alpha}{2-\alpha}})$ at each step.

Note that $D\phi_a$ and $D^2\phi_a$ still satisfy inequalities (A.12) and (A.16) in Lemma A.5. We will choose a such that inequality (A.16) takes the form of (2.1), i.e. we choose a such that $\frac{\varepsilon}{a}\eta^3 \leqslant K$. Since $\eta^2 \leqslant 1$ we choose $a = \varepsilon$. Again we use the definition of viscosity solutions and subtract the equations (inequalities) at $\bar{x}$ and $\bar{y}$ using the maximum principle for semicontinuous functions. By the appropriate version of Lemma A.5 and the definition of η^2 and B we can now use (H1) and $(\overline{\text{H2}})$ to get

$$\lambda\big(u(\bar{x}) - u(\bar{y})\big) \leqslant K\left(|\bar{x}-\bar{y}|^\alpha + \frac{1}{\varepsilon^2}|\bar{x}-\bar{y}|^2 + \eta^2 + \varepsilon^2 + K_B\big(\varepsilon + \varepsilon^{\frac{2\alpha}{2-\alpha}}\big)\right).$$

By Young's inequality, the definition of η^2, and $\varepsilon \leqslant 1$ we have

$$\lambda\big(u(\bar{x}) - u(\bar{y})\big) \leqslant K\left(\frac{1}{\varepsilon^2}|\bar{x}-\bar{y}|^2 + \varepsilon^{1\wedge\frac{2\alpha}{2-\alpha}}\right).$$

When A is big enough, an appropriate version of Lemma A.3, the definition of M, and $0 \leqslant d \leqslant 1$, imply that

$$u(\bar{x}) - u(\bar{y}) = M + \phi(\bar{x},\bar{y}) \geqslant M + \frac{1}{2\varepsilon^2}e^{-2K_e}|\bar{x}-\bar{y}|^2 - K_0\varepsilon^2 - K_B\big(\varepsilon + \varepsilon^{\frac{2\alpha}{2-\alpha}}\big)\big(d(\bar{x}) + d(\bar{y})\big).$$

Combining the two last inequalities and using that $\varepsilon \leqslant 1$ leads to

$$\lambda M \leqslant \left(K - \frac{\lambda}{2}e^{-2K_e}\right)\frac{1}{\varepsilon^2}|\bar{x}-\bar{y}|^2 + K\varepsilon^{1\wedge\frac{2\alpha}{2-\alpha}}.$$

If λ is big enough, $\lambda M \leqslant K\varepsilon^{1\wedge\frac{2\alpha}{2-\alpha}}$, and the definition of M leads to

$$u(x) - u(y) - \phi_\varepsilon(x,y) \leqslant M \leqslant \frac{K}{\lambda}\varepsilon^{1\wedge\frac{2\alpha}{2-\alpha}}$$

for every $x, y \in \bar{\Omega}$. Now by the definition of ϕ_a, the properties of the distance function, and Young's inequality, we have

$$u(x) - u(y) \leqslant K\frac{1}{\varepsilon^2}|x-y|^2 + K\varepsilon^{1\wedge\frac{2\alpha}{2-\alpha}}.$$

If $|x-y| \leqslant 1$ we may take $\varepsilon = |x-y|^{\frac{2}{3}}$ when $1 < \frac{2\alpha}{2-\alpha}$ and $\varepsilon^{\frac{2\alpha}{2-\alpha}} = |x-y|^{\alpha}$ otherwise, the result (since we may also interchange x and y) is that

$$\left|u(x) - u(y)\right| \leqslant K|x-y|^{\frac{2}{3}\wedge\alpha}. \tag{3.5}$$

If $|x-y| \geqslant 1$, the result still holds since then $|u(x)-u(y)| \leqslant 2|u|_0|x-y|^{\frac{2}{3}\wedge\alpha}$. We are now done if $\alpha \leqslant \frac{2}{3}$.

If $\alpha \in (\frac{2}{3}, 1]$, we restart the proof using the regularity estimate (3.5) to get a better choice of a such that $\frac{\varepsilon}{a}\eta^3 \leqslant K$. From (3.5) and the first inequality in (3.4),

$$\eta^2 = K^{-1}\frac{1}{\varepsilon^2}|\bar{x}-\bar{y}|^2 \leqslant K|\bar{x}-\bar{y}|^{\frac{2}{3}} \vee \varepsilon^2 \quad \text{and hence} \quad \eta \leqslant K\varepsilon^{\frac{\frac{2}{3}}{2-\frac{2}{3}}} \vee \varepsilon,$$

so the new choice of a should be $\varepsilon\eta^3 = \varepsilon^{\frac{5}{2}} \vee \varepsilon^4$. But this quantity is less than ε^2 so we may instead take $a = \varepsilon^2$ which still implies $\frac{\varepsilon}{a}\eta^3 \leqslant K$. Now it is a simple exercise to redo the proof and show that for λ big,

$$\left|u(x) - u(y)\right| \leqslant K|x-y|^{\alpha} \quad \text{for } x, y \in \bar{\Omega},$$

and this completes the proof of the last part of Theorem 2.1.

Now we will prove the first part of Theorem 2.1(c) using the result we proved above and an iterative argument of Lions [27]. Here we only sketch parts of the argument, since the details can be found in [21] for similar equations. The idea is to consider for $\mu > 0$

$$F\left(x, u^{n+1}, Du^{n+1}, D^2u^{n+1}\right) + \mu u^{n+1} = \mu u^n$$

with boundary conditions (1.2) and noting that u^n converge uniformly to u. If μ is big enough the above proven result applies, and a careful look at the above argument reveals that when $\lambda + \mu > K(=\bar{\lambda})$, then

$$\left|u^{n+1}(x) - u^{n+1}(y)\right| \leqslant \left(\frac{\mu|u^n|_\alpha}{\lambda+\mu-K} + \text{other terms}\right)|x-y|^{\alpha}, \quad x, y \in \bar{\Omega}, |x-y| \leqslant 1.$$

Furthermore the comparison principle yields

$$\left|u^{n+1} - u\right|_0 \leqslant \frac{\mu}{\mu+\lambda}\left|u^n - u\right|_0 \leqslant \left(\frac{\mu}{\mu+\lambda}\right)^n\left|u^0 - u\right|_0.$$

When $|x-y| \leqslant 1$, the rest of the proof is exactly as in [21] and we omit it. When $|x-y| > 1$ any Hölder estimate holds since u is bounded. The result is a Hölder estimate for any $\lambda > 0$, but with a Hölder exponent that is smaller than α. □

4. Bellman–Isaacs type boundary value problems

In this section we apply our results in Section 2 to Bellman–Isaacs equations and several different types of boundary conditions. The Bellman–Isaacs equations are of the form

$$\inf_{\theta_1\in\Theta_1}\sup_{\theta_2\in\Theta_2}\left\{-\operatorname{tr}\left[\left(\sigma\sigma^T\right)^{\theta_1,\theta_2}(x)D^2u\right]-b^{\theta_1,\theta_2}(x)Du-c^{\theta_1,\theta_2}(x)u-f^{\theta_1,\theta_2}(x)\right\}=0 \tag{4.1}$$

in Ω. Assumptions (H1), (H2), ($\overline{\text{H2}}$), and (H3) are satisfied [12,21] if we assume:

1. $\sigma^{\theta_1,\theta_2}$ and b^{θ_1,θ_2} are Lipschitz continuous in x uniformly in θ_1,θ_2,
2. c^{θ_1,θ_2} and f^{θ_1,θ_2} are α-Hölder continuous in x uniformly in θ_1,θ_2,
3. $c^{\theta_1,\theta_2}(x)\geqslant\lambda>0$ for all x,θ_1,θ_2, and
4. Θ_1,Θ_2 are compact metric spaces.

Next, we list some typical boundary conditions we can consider:

(a) The classical Neumann condition:

$$\frac{\partial u}{\partial n}=g(x)\quad\text{in }\partial\Omega.$$

(b) The oblique derivative condition:

$$\frac{\partial u}{\partial\gamma}=g(x)\quad\text{in }\partial\Omega.$$

(c) The capillary boundary condition:

$$\frac{\partial u}{\partial n}=\bar\theta(x)\left(1+|Du|^2\right)^{1/2}\quad\text{in }\partial\Omega\text{ with }\left|\bar\theta(x)\right|\leqslant\omega<1. \tag{4.2}$$

(d) The "controlled" reflection boundary condition:

$$\inf_{\alpha\in\Theta_1}\sup_{\beta\in\Theta_2}\left\{\gamma^{\theta_1,\theta_2}(x)\cdot Du-g^{\theta_1,\theta_2}(x)\right\}=0\quad\text{in }\partial\Omega. \tag{4.3}$$

Here $n(x)$ is the outward unit normal to $\partial\Omega$. Assumptions (HB1) and (HB2) hold in all cases if assumption 4 holds along with:

5. There exists $\nu>0$ such that

$$\gamma(x)\cdot n(x)\geqslant\nu\quad\text{and}\quad\gamma^{\theta_1,\theta_2}(x)\cdot n(x)\geqslant\nu\quad\text{uniformly in }\theta_1,\theta_2.$$

6. $g,\gamma,\bar\theta,\gamma^{\theta_1,\theta_2},g^{\theta_1,\theta_2}$ are Lipschitz continuous in x uniformly in θ_1,θ_2.

Now we state new continuous dependence results for the for the Bellman–Isaacs equations (4.1) combined with the controlled reflection boundary conditions (4.3):

Theorem 4.1. *Assume u_1 and u_2 satisfy the boundary value problem* (4.1) *and* (4.3) *with coefficients $\sigma_1, b_1, c_1, f_1, \gamma_1, g_1$ and $\sigma_2, b_2, c_2, f_2, \gamma_2, g_2$ respectively, where both sets of coefficients satisfy assumptions* 1–6 *above.*

Then u_1, u_2 belong to $C^{0,\beta}(\bar{\Omega})$ for some $\beta \in (0, \alpha]$, and

$$\begin{aligned}\lambda|u_1 - u_2|_0 \leqslant{} & C \sup_{\Theta_1 \times \Theta_2} \big[\big|\sigma_1^{\theta_1,\theta_2} - \sigma_2^{\theta_1,\theta_2}\big|_0^\beta + \big|b_1^{\theta_1,\theta_2} - b_2^{\theta_1,\theta_2}\big|_0^\beta\big] \\ & + C \sup_{\Theta_1 \times \Theta_2} \big[\big|c_1^{\theta_1,\theta_2} - c_2^{\theta_1,\theta_2}\big|_0 + \big|f_1^{\theta_1,\theta_2} - f_2^{\theta_1,\theta_2}\big|_0\big] \\ & + \frac{C}{\nu} \sup_{\Theta_1 \times \Theta_2} \big|g_1^{\theta_1,\theta_2} - g_2^{\theta_1,\theta_2}\big|_0 + \frac{C}{\nu^\beta} \sup_{\Theta_1 \times \Theta_2} \big|\gamma_1^{\theta_1,\theta_2} - \gamma_2^{\theta_1,\theta_2}\big|_0^\beta.\end{aligned}$$

This result is a direct consequence of Theorems 2.1 and 2.2. In this case δ_1 corresponds to the second line in the estimate,

$$\delta_2^2 = C \sup_{\Theta_1 \times \Theta_2} \big[\big|\sigma_1^{\theta_1,\theta_2} - \sigma_2^{\theta_1,\theta_2}\big|^2 + \big|b_1^{\theta_1,\theta_2} - b_2^{\theta_1,\theta_2}\big|^2\big],$$

$$\mu_1 = \sup_{\Theta_1 \times \Theta_2} \big|g_1^{\theta_1,\theta_2} - g_2^{\theta_1,\theta_2}\big|_0, \qquad \mu_2 = \sup_{\Theta_1 \times \Theta_2} \big|\gamma_1^{\theta_1,\theta_2} - \gamma_2^{\theta_1,\theta_2}\big|_0.$$

The dependence on the equation is as in [20,21] and the derivation of δ_1 and δ_2 is explained there.

By Theorem 2.3 we have for the first time the rate of convergence of the vanishing viscosity method for the boundary value problem (4.1) and (4.3), i.e.

$$\inf_{\theta_1 \in \Theta_1} \sup_{\theta_2 \in \Theta_2} \big\{-\operatorname{tr}\big[\big(\sigma\sigma^T\big)^{\theta_1,\theta_2}(x) D^2 u\big] - b^{\theta_1,\theta_2}(x) Du - c^{\theta_1,\theta_2}(x) u - f^{\theta_1,\theta_2}(x)\big\} = \mu \Delta u \tag{4.4}$$

in Ω, with (4.3) as boundary conditions. The result is the following:

Theorem 4.2. *Assume u and u_μ satisfy* (4.1) *and* (4.4) *respectively with boundary values* (4.3)*, and that assumptions* 1–6 *hold.*

Then u, u_μ belong to $C^{0,\beta}(\bar{\Omega})$ for some $\beta \in (0, \alpha]$ and

$$|u - u_\mu|_0 \leqslant C\mu^{\frac{\beta}{2}}.$$

5. Extensions

It is possible to consider many kinds of extensions of the results in this paper. We will consider three cases: (i) Ω unbounded, (ii) time-dependent problems, and (iii) quasilinear equations. In the two first cases the results cover e.g. Bellman–Isaacs equations under natural assumptions on the data.

5.1. Unbounded domains

Let Ω be unbounded and let (H0u) denote assumption (H0) without the boundedness assumption. If we assume that our sub- and supersolutions u and v are bounded, then we will get continuous dependence and regularity results simply by following the arguments in this paper replacing the test function ϕ_a by the standard modification

$$\phi_a(x, y) + \gamma\big(|x|^2 + |y|^2\big), \quad \gamma > 0.$$

The new test function will insure existence of maximum points when we double the variables, and at the end of the proof it turns out (as usual) that all terms depending on γ will vanish when $\gamma \to 0$. In the proof B will now depend also on the γ-terms and the γ-terms will tend to zero as $\gamma \to 0$ with a speed depending on B, see assumption (D1u) below. By careful computations and fixing ε before sending $\gamma \to 0$ we can conclude as before. We refer to [21] for the details when $\Omega = \mathbb{R}^N$.

The corresponding continuous dependence result will now be given without further proof. We modify assumption (D1) so it corresponds to our new test function, see also [21]:

(D1u) There are $\delta_1, \delta_2 \geqslant 0$, a modulus ω, and $K_F(K) \geqslant 0$, such that for any $K \geqslant 0$,

$$F_2(y, r, q, Y) - F_1(x, r, p, X) \leqslant K_F(K)\left(\eta^2 + \delta_1 + \frac{1}{\varepsilon^2}\delta_2^2 + B + \gamma\big(1 + |x|^2 + |y|^2\big)\right),$$

for $\varepsilon, \gamma \in (0, 1]$, $\eta := \varepsilon^{\frac{\alpha}{2-\alpha}}$, $B \geqslant 0$, $x, y \in \bar{\Omega}$, $r \in \mathbb{R}$, $|r| \leqslant K$, $p, q \in \mathbb{R}^N$ and $X, Y \in \mathbb{S}^N$ satisfying $|x - y| \leqslant K\eta\varepsilon$, $|x| + |y| \leqslant \gamma^{1/2}\omega(\gamma)(1 + B)$, $|p - q| \leqslant K(\eta^2 + B + \gamma(|x| + |y|))$, $|p| + |q| \leqslant K(\frac{\eta}{\varepsilon} + \eta^2 + B + \gamma(|x| + |y|))$, and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant \frac{K}{\varepsilon^2}\begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{pmatrix} + K\big(\eta^2 + B + \gamma\big)\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}.$$

Theorem 5.1 *(Ω unbounded). Assume* (H0u)*,* (H1)*,* (H3)*,* (HB1)*, and* (HB2) *hold for H_1, H_2, G_1, G_2, and $u_1, u_2 \in C^{0,\beta}(\bar{\Omega})$ for $\beta \in (0, 1]$. Define $\nu^2 = (\nu_1 \vee \nu_2)(\nu_1 \wedge \nu_2)$ and $\lambda = \lambda_{1,|u_1|_0} \vee \lambda_{2,|u_2|_0}$.*

If (D1u) *and* (D2) *hold and u_1 and u_2 satisfy the boundary value problems* (2.2) *and* (2.3) *respectively then there exists a constant $C > 0$ (depending only on K_F, K, K_G, $|u_1|_\beta$, $|u_2|_\beta$, α) such that*

$$\lambda \max_{\bar{\Omega}}(u_1 - u_2) \leqslant C\left(\delta_1 + \delta_2^{\alpha\wedge\beta} + \frac{\mu_1}{\nu} + \left(\frac{\mu_2}{\nu}\right)^{\alpha\wedge\beta}\right).$$

5.2. Time-dependent case

Consider a Cauchy–Neumann problem of the form:

$$u_t + F\big(t, x, u, Du, D^2u\big) = 0 \qquad \text{in } (0, T) \times \Omega, \tag{5.1}$$

$$G(x, Du) = 0 \qquad \text{on } (0, T) \times \partial\Omega, \tag{5.2}$$

$$u(0, x) = u_0(x) \qquad \text{on } \{0\} \times \Omega. \tag{5.3}$$

In this case we get results by similar arguments as above by replacing the test function ϕ_a by

$$\bar{\sigma} t + e^{Kt}\phi_a(x, y), \quad \bar{\sigma} > 0.$$

We have to replace assumptions (H1)–(H3) and (D1) by assumptions (H1p)–(H3p) and (D1p) depending on t. In (H1p) we assume in addition continuity in t, in (H3p) we allow $\lambda_R \geqslant 0$, and in the last two assumptions ((H2p) and (D1p)) we simply assume that (H2) and (D1) hold uniformly in t. Note that one can always reduce a problem with $\lambda_R \in \mathbb{R}$, via an exponential scaling of u, to a problem with $\lambda_R \geqslant 0$.

Now existence, uniqueness, and regularity results follow as before by appropriately choosing the constants $\bar{\sigma}$ and K. Note however that in the result corresponding to Theorem 2.1(c) the Hölder exponent is always α and "maximal regularity" is achieved regardless of the value λ. We refer to [20] for such results in the case $\Omega = \mathbb{R}^N$. Now we state the continuous dependence result without further proof.

Theorem 5.2 *(Time-dependent case). Assume* (H0)*,* (H1p)*,* (H3p)*,* (HB1)*, and* (HB2) *hold for* H_1, H_2, G_1, G_2*,* $u_1, u_2 \in C([0,T] \times \bar{\Omega})$*, and* $u_{1,0}, u_{2,0} \in C^{0,\alpha}(\bar{\Omega})$ *for some* $\alpha \in (0,1]$*. Define* $\nu^2 = (\nu_1 \vee \nu_2)(\nu_1 \wedge \nu_2)$*.*

If (D1p) *and* (D2) *hold and* u_1 *and* u_2 *are sub- and supersolutions of initial boundary value problems* (5.1)*,* (5.2)*, and* (5.3) *respectively for* $F_1, G_1, u_{1,0}$ *and* $F_2, G_2, u_{2,0}$*, then there exists a constant* $C > 0$ (*depending only on* K_F*,* K*,* K_G*,* $|u_{1,0}|_\alpha$*,* $|u_{2,0}|_\alpha$*,* T*,* α) *such that for* $t \in (0,T)$*,*

$$\max_{\bar{\Omega}}\big(u_1(t,\cdot) - u_2(t,\cdot)\big) \leqslant \big|(u_{1,0} - u_{2,0})^+\big|_0 + Ct\left(\delta_1 + \delta_2^\alpha + \frac{\mu_1}{\nu} + \left(\frac{\mu_2}{\nu}\right)^\alpha\right).$$

Note that we do not need to assume that u_1 and u_2 are Hölder continuous (in x) a priori. In fact this regularity follows from the above theorem! To understand why, and to see details about the derivation in the case $\Omega = \mathbb{R}^N$, we refer to [20].

5.3. Some quasilinear equations

Consider equations of the form

$$-\operatorname{tr}\big[\sigma(x,Du)\sigma(x,Du)^T D^2u\big] - f(x,u,Du) + \lambda u = 0 \quad \text{in } \Omega, \tag{5.4}$$

where $\lambda > 0$, $f(x,r,p)$ continuous, increasing in r, $a(x,p) = \sigma(x,p)\sigma(x,p)^T$, and

$$\big|\sigma(x,p) - \sigma(y,q)\big| \leqslant K\left(|x-y| + \frac{|p-q|}{1+|p|+|q|}\right),$$
$$\big|f(x,r,p) - f(y,r,q)\big| \leqslant K\big[\big(1+|p|+|q|\big)|x-y| + |p-q|\big].$$

In this case (H1) and (H3) hold in addition to an assumption similar to (H2). If we also assume (H0), (HB1), and (HB2), then existence and comparison for the boundary value problem (5.4) and (1.2) was proved in [2].

More general fully non-linear equations with "quasilinear" gradient dependence can also be considered. We omit this to get a shorter and clearer presentation. For the same reasons we also restrict ourselves to the case of Lipschitz continuous solutions and data, i.e. $\alpha = \beta = 1$, $\eta \equiv \varepsilon$.

In this case the quasilinear term in the equation gives rise to a term like

$$\frac{1}{\varepsilon^2}\big|\sigma(p) - \sigma(q)\big|^2$$

in the proof of the comparison result (when σ does not depend on x). By (A.15) in Lemma A.5,

$$|p-q| \leqslant K|p| \wedge |q||x-y| + K\big(\varepsilon^2 + B\big)$$

when ε is small enough, and hence by the assumptions on σ,

$$\begin{aligned}\frac{1}{\varepsilon^2}\big|\sigma(p)-\sigma(q)\big|^2 &\leqslant \frac{K}{\varepsilon^2}|x-y|^2 + \frac{K}{\varepsilon^2}\frac{\varepsilon^4+B^2}{1+|p|^2+|q|^2} \\ &\leqslant \frac{K}{\varepsilon^2}|x-y|^2 + K\varepsilon^2 + \frac{K}{\varepsilon^2}B^2. \end{aligned} \tag{5.5}$$

This computation motivates replacing assumption (D1) by:

(D1q) There are $\delta_1, \delta_2 \geqslant 0$, and $K_F(K) \geqslant 0$ such that for any $K \geqslant 0$,

$$F_2(y,r,q,Y) - F_1(x,r,p,X) \leqslant K_F(K)\left(\eta^2 + \delta_1 + \frac{1}{\varepsilon^2}\delta_2^2 + B + \frac{1}{\varepsilon^2}B^2\right),$$

for $0 < \varepsilon \leqslant 1$, $B \geqslant 0$, $x, y \in \bar{\Omega}$, $r \in \mathbb{R}$, $|r| \leqslant K$, $p, q \in \mathbb{R}^N$ and $X, Y \in \mathbb{S}^N$ satisfying $|x-y| \leqslant K\varepsilon^2$, $|p-q| \leqslant K|p| \wedge |q|\varepsilon^2 + K(\varepsilon^2 + B)$, $|p| + |q| \leqslant K(1 + \varepsilon^2 + B)$, and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant \frac{K}{\varepsilon^2}\begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{pmatrix} + K\big(\varepsilon^2 + B\big)\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}.$$

The continuous dependence result now becomes:

Theorem 5.3 *(Quasilinear equations, Lipschitz solutions). Assume* (H0), (H1), (H3), (HB1), *and* (HB2) *hold for H_1, H_2, G_1, G_2, and $u_1, u_2 \in C^{0,1}(\bar{\Omega})$. Define $\nu^2 = (\nu_1 \vee \nu_2)(\nu_1 \wedge \nu_2)$.*

If (D1q) *and* (D2) *hold and u_1 and u_2 are sub- and supersolutions of the boundary value problems* (2.2) *and* (2.3) *respectively, then there exists a constant $C > 0$ (depending only on K_F, K, K_G) such that*

$$\lambda \max_{\bar{\Omega}}(u_1 - u_2) \leqslant C\left(\delta_1 + \delta_2 + \frac{\mu_1}{\nu} + \frac{\mu_2}{\nu}\right).$$

Proof. The proof is similar to the proof of Theorem 2.2 with two exceptions:

(i) Assume $\delta_1, \delta_2, \frac{\mu_1}{\nu}, \frac{\mu_2}{\nu} \leqslant \bar{C}^{-1}$ where $\bar{C}$ is big enough (the general case follows since u_1, u_2 are bounded). Since ε is chosen in terms of $\delta_1, \delta_2, \frac{\mu_1}{\nu}, \frac{\mu_2}{\nu}$, a suitable choice of $\bar{C}$ will ensure that ε is small enough such that (A.15) of Lemma A.5 holds. This estimate is needed before one can apply (D1q).

(ii) At the end of the proof the following estimate will appear (remember $\eta = \varepsilon$)

$$\lambda_1\big(u_1(x) - u_2(x)\big) \leqslant K\left(\eta^2 + \delta_1 + \frac{1}{\varepsilon^2}\delta_2^2 + \frac{[\mu_1 + \mu_2\frac{\eta}{\varepsilon}]}{\nu} + \frac{1}{\varepsilon^2}\left(\frac{[\mu_1 + \mu_2\frac{\eta}{\varepsilon}]}{\nu}\right)^2\right),$$

where the *new final term* in the right-hand side of the inequality is a consequence of the $\frac{1}{\varepsilon^2}B^2$ term of (D1q). Minimizing ε like we did in Theorem 2.2 then gives the result. □

As an example we consider an anisotropic quasilinear equation with capillary boundary condition. The type of non-linearity appearing here is similar to the non-linearity appearing in the mean curvature of graph equation.

$$-\operatorname{tr}\left[\sigma\sigma^T\left(I-\frac{Du\otimes Du}{1+|Du|^2}\right)D^2u\right]+\lambda u+f(x,u,Du)=0\quad\text{in }\Omega,$$
$$\frac{\partial u}{\partial n}-\bar\theta(x)\left(1+|Du|^2\right)^{1/2}=0\quad\text{in }\partial\Omega,$$

where $\bar\theta$ is Lipschitz continuous satisfying $|\bar\theta(x)|\leqslant\omega<1$ and f satisfies the assumptions mentioned above. Assume u_1 and u_2 are Lipschitz solutions of this boundary value problem with different $\sigma_1,\sigma_2,\theta_1,\theta_2$ but with the same f and λ. Then we may apply Theorem 5.3 with $\delta_1=0=\mu_1$, $\mu_2=|\bar\theta_1-\bar\theta_2|_0$, and

$$\delta_2^2=\sup_{p\in\mathbb{R}^N}\left|\sqrt{\sigma_1\sigma_1^T\left(I-\frac{p\otimes p}{1+|p|^2}\right)}-\sqrt{\sigma_2\sigma_2^T\left(I-\frac{p\otimes p}{1+|p|^2}\right)}\right|^2,$$

to obtain

$$\lambda|u_1-u_2|\leqslant C\left(|\sigma_1-\sigma_2|+\frac{1}{\nu}|\bar\theta_1-\bar\theta_2|_0\right).$$

Remark 5.1. Neither assumptions (5.5) nor assumption (D1q) is satisfied for p-Laplacian equations.

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Appendix A. The test function ϕ_a

A.1. Construction

The construction of the test function follows G. Barles [1,2] tracking some quantities more precisely than he needs to. We give some more details (compared to [1,2]) whenever we feel this is helpful for the reader.

First let d be a $W^{3,\infty}$ extension of the (signed) distance function from some neighborhood of $\partial\Omega$ to $\mathbb{R}^N$ (by (H0) the distance function is $W^{3,\infty}$ near $\partial\Omega$). Furthermore we may and will choose d such that $0\leqslant d\leqslant 1$ and $|Dd(x)|\leqslant 1$ in $\bar\Omega$. We also extend the outward normal vector field n of $\partial\Omega$ to all of $\mathbb{R}^N$ by setting $n(x):=-Dd(x)$ for $x\in\Omega$. An important consequence of the $W^{3,\infty}$ regularity of d and Taylor's Theorem, are the following inequalities:

$$\pm\left[d(x)-d(y)\right]\leqslant\pm(y-x)\cdot n\left(\frac{x+y}{2}\right)+\frac{1}{24}\left|D^3d\right|_0|x-y|^3.\tag{A.1}$$

Next we extend $G_i, i = 1, 2$, to a neighborhood V of $\partial\Omega$ such that properties (HB1) and (HB2) still hold here (possibly with different constants K and ν). Then (HB1) and the intermediate value theorem that implies that there exists unique solutions $C_i(x, p)$, of

$$G_i\big(x, p + C_i(x, p)n(x)\big) = 0, \quad i = 1, 2, \tag{A.2}$$

for every x (in the neighborhood V) and p. Note that

$$G_i(x, p) - G_i\big(x, p + C_i(x, p)\big) = G_i(x, p) = G_i(x, 0) + \big(G_i(x, p) - G_i(x, 0)\big)$$

and

$$\begin{aligned} &G_i\big(x, p + C_i(y, q)n(x)\big) - G_i\big(x, p + C_i(x, p)n(x)\big) \\ &\quad = G_i\big(x, p + C_i(y, q)n(x)\big) = G_i\big(x, p + C_i(y, q)n(x)\big) - G_i\big(y, q + C_i(y, q)n(y)\big), \end{aligned}$$

so by (HB1) and (HB2)

$$\nu_i\big|C_i(x, p)\big| \leqslant K\big(1 + |p|\big) \tag{A.3}$$

and

$$\nu_i\big|C_i(x, p) - C_i(y, q)\big| \leqslant K\big(1 + |p| + |q|\big)|x - y| + K|p - q|, \tag{A.4}$$

for $x, y \in V$, $p, q \in \mathbb{R}^N$. Now we extend $C_i, i = 1, 2$, from V to $\mathbb{R}^N$ (in the x variable) such that (A.4) and (A.3) are preserved (possibly with bigger K's). Next note that

$$\begin{aligned} &G_1\big(x, p + C_1(x, p)n(x)\big) - G_1\big(x, p + C_2(x, p)n(x)\big) \\ &\quad = -G_1\big(x, p + C_2(x, p)n(x)\big) = G_2\big(x, p + C_2(x, p)n(x)\big) - G_1\big(x, p + C_2(x, p)n(x)\big). \end{aligned}$$

Therefore, for $x \in \partial\Omega$, by assumptions (HB1) and (D2) we get

$$\nu_1\big(C_1(x, p) - C_2(x, p)\big) \leqslant K_G\big(\mu_1 + \mu_2\big[|p| + K\big(1 + |p|\big)\big]\big).$$

We have proved:

Lemma A.1. *Assume* (H0), (HB1), *and* (HB2).

(a) *There exist unique functions C_1 and C_2 such that Eq.* (A.2) *holds in a neighborhood of $\partial\Omega$, and the bounds* (A.3) *and* (A.4) *hold for all $x, p \in \mathbb{R}^N$.*

(b) *If in addition* (D2) *holds, then there exists a constant $K_C > 0$ such that for every $p \in \mathbb{R}^N$ and $x \in \partial\Omega$,*

$$C_1(x, p) - C_2(x, p) \leqslant \frac{K_C}{\nu_1 \vee \nu_2}\big(\mu_1 + \mu_2\big(1 + |p|\big)\big). \tag{A.5}$$

To build smooth functions from C_1 and C_2 we will use the following sophisticated regularization due to G. Barles [2]:

Definition A.1. Let $a > 0$. For a function $C(x, p)$ on $\mathbb{R}^N \times \mathbb{R}^N$ we define

$$C_a(x,p) := \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} C(y,q)\rho\left((x-y)\frac{\Gamma}{\Lambda}\right)\rho\left(\frac{p-q}{\Lambda}\right)\frac{\Gamma^N}{\Lambda^{2N}}\,dy\,dq,$$

where

$$\Lambda = \left(a^2 + \left(p \cdot n(x)\right)^2\right)^{\frac{1}{2}} \quad \text{and} \quad \Gamma = \left(1 + |p|^2\right)^{\frac{1}{2}},$$

and $\rho \in C^\infty(\mathbb{R}^N)$ is a nonnegative function with total mass 1 and support in $|x| \leqslant 1$.

Lemma A.2. *If $C(x, p)$ satisfies* (A.4) *for $x, p \in \mathbb{R}^N$ and $0 < a \leqslant 1$, then for any $x, p \in \mathbb{R}^N$,*

$$\left|C_a(x,p)\right| \leqslant K\Gamma \quad \textit{and} \quad \left|C_a(x,p) - C(x,p)\right| \leqslant K\left(a + \left|p \cdot n(x)\right|\right), \tag{A.6}$$

$$\left|D_x C_a(x,p)\right| \leqslant K\Gamma, \qquad \left|D_p C_a(x,p)\right| \leqslant K, \tag{A.7}$$

$$\left|D_{xx} C_a(x,p)\right| \leqslant K\frac{\Gamma^2}{\Lambda}, \qquad \left|D_{xp} C_a(x,p)\right| \leqslant K\frac{\Gamma}{\Lambda}, \qquad \left|D_{pp} C_a(x,p)\right| \leqslant \frac{K}{\Lambda}. \tag{A.8}$$

The proof follows from the classical properties of convolution and the regularity of C (A.4) together with the choice of Λ and Γ.

Now remember that d is a $W^{3,\infty}$ extension of the distance function, and let $C_{2,a}$ be the smooth function obtained from Definition A.1 with $C = C_2$. The full test function takes the following form:

Definition A.2 *(The test function ϕ_a).*

$$\phi_a(x,y) = \frac{1}{\varepsilon^2}|x-y|^2 + \frac{A}{\varepsilon^2}\left(d(x) - d(y)\right)^2 - B\left(d(x) + d(y)\right)$$
$$- C_{2,a}\left(\frac{x+y}{2}, \frac{2(x-y)}{\varepsilon^2}\right)\left(d(x) - d(y)\right),$$

where $A, B \geqslant 0$ are constants.

A.2. Properties

In the next three lemmas we state the main properties of the test function ϕ_a that we need in this paper.

Lemma A.3 *(Positivity). Assume* (H0)*,* (HB1)*,* (HB2)*, and let ϕ be defined in Definition* A.2*. If A is big enough* (*not depending on ε, a, B*)*, then*

$$\phi_a(x,y) \geqslant \frac{1}{2\varepsilon^2}|x-y|^2 - K_0\varepsilon^2 - B\left(d(x) + d(y)\right).$$

Proof. By (A.6), $|C_{2,a}(x,p)| \leqslant C(1+|p|)$ with C independent of a. So by Young's inequality one can take A big enough to insure that

$$\frac{1}{2\varepsilon^2}|x-y|^2 + \frac{A}{2\varepsilon^2}\big(d(x)-d(y)\big)^2 - C_{2,a}\left(\frac{x+y}{2}, \frac{2(x-y)}{\varepsilon^2}\right)\big(d(x)-d(y)\big) \geqslant -K_0\varepsilon^2,$$

which proves the lemma. □

Lemma A.4 *(Boundary conditions). Assume* (H0), (HB1), (HB2), (D2), $0<\varepsilon,\eta,a \leqslant 1$, *and let* ϕ_a *be defined in Definition* A.2. *Then for any* $x, y \in \bar{\Omega}$ *such that* $|x-y| \leqslant K_1\eta\varepsilon$, *there exists a* $K \geqslant 0$ *only depending on the data and on* K_1, *such that if*

$$B = K\big(\eta^2+\varepsilon^2+a\big) + \frac{K}{\nu_1 \vee \nu_2}\left(\mu_1 + \mu_2\frac{\eta}{\varepsilon}\right) \quad \textit{and} \quad A = K,$$

then

$$G_1\big(x, D_x\phi_a(x,y)\big) > 0 \quad \textit{if } x \in \partial\Omega, \tag{A.9}$$

$$G_2\big(x, -D_y\phi_a(x,y)\big) < 0 \quad \textit{if } y \in \partial\Omega. \tag{A.10}$$

Proof. We only prove (A.9), the proof of (A.10) is similar but easier due to the choice of $C_{2,a}$ in the test function. Note that $d(x)=0$, $d(y)-d(x) = |d(x)-d(y)|$, and remember that $n=-Dd$. We have

$$\begin{aligned}
D_x\phi_a(x,y) &= \frac{2(x-y)}{\varepsilon^2} + \left[\frac{1}{2}D_xC_{2,a}(X,p) + \frac{2}{\varepsilon^2}D_pC_{2,a}(X,p)\right]\big(d(y)-d(x)\big) \\
&\quad + \left[C_{2,a}(X,p) + B + \frac{2A}{\varepsilon^2}\big(d(y)-d(x)\big)\right]n(x) \\
&= p + R_1 + \big[C_1(x,p)+R_2\big]n(x),
\end{aligned} \tag{A.11}$$

where $p = \frac{1}{\varepsilon^2}|x-y|$, $X = \frac{1}{2}(x+y)$, and

$$R_2 := \big[-C_1(x,p)+C_2(x,p)\big] + \big[-C_2(x,p)+C_{2,a}(X,p)\big] + B + \frac{2A}{\varepsilon^2}\big(d(y)-d(x)\big).$$

According to (A.6), (A.7) and (A.1), the second term in R_2 is bounded below by

$$-K\big(1+|p|\big)\frac{|x-y|}{2} - K\left(a + \left|p\cdot n\left(\frac{x+y}{2}\right)\right|\right) \geqslant -K\left(\eta^2+\varepsilon^2+a+\frac{|d(x)-d(y)|}{\varepsilon^2}\right).$$

Here we have also used that $p=\frac{2}{\varepsilon^2}|x-y|$ and $|x-y| \leqslant K\varepsilon\eta$.

According to (A.5), the first term in R_2 is bounded from below by

$$-\frac{K_C}{\nu_1 \vee \nu_2}\big(\mu_1+\mu_2\big(1+|p|\big)\big) \geqslant -\frac{K}{\nu_1 \vee \nu_2}\left(\mu_1+\mu_2\frac{\eta}{\varepsilon}\right).$$

This means that $R_2 > 0$ if $B > K(\eta^2 + \varepsilon^2 + a) + \frac{K}{\nu_1 \vee \nu_2}(\mu_1 + \mu_2 \frac{\eta}{\varepsilon})$ and $A \geqslant \frac{K}{2}$. By (A.7) of Lemma A.2, the regularity of d, $|x-y| \leqslant K\varepsilon\eta$, we also find that

$$|R_1| \leqslant \left[K\big(1+|p|\big) + \frac{1}{\varepsilon^2}K\right]\big(d(y)-d(x)\big) \leqslant K\left(\eta^2 + \varepsilon^2 + \frac{|d(x)-d(y)|}{\varepsilon^2}\right).$$

By (HJB1), (HJB2), (A.11), (A.2), and the above estimates and choices of A, B,

$$\begin{aligned} G_1\big(x, D_x\phi_a(x,y)\big) &\geqslant G_1\big(x, p + C_1(x,p)\big) + \nu_1 R_2 - K|R_1| \\ &\geqslant 0 + \nu_1\bigg[B - K\big(\eta^2+\varepsilon^2+a\big) - \frac{K}{\nu_1 \vee \nu_2}\left(\mu_1 + \mu_2\frac{\eta}{\varepsilon}\right) \\ &\qquad + (2A-K)\frac{|d(x)-d(y)|}{\varepsilon^2}\bigg] - K\left(\eta^2+\varepsilon^2+\frac{|d(x)-d(y)|}{\varepsilon^2}\right), \end{aligned}$$

and the right-hand side is strictly positive if

$$\nu_1 B = (1+\nu_1)K\big(\eta^2+\varepsilon^2+a\big) + \frac{\nu_1 K}{\nu_1 \vee \nu_2}\left(\mu_1 + \mu_2\frac{\eta}{\varepsilon}\right) \quad \text{and} \quad 2\nu_1 A = (1+\nu_1)K. \qquad \square$$

Lemma A.5 *(Derivatives). Assume* (H0), (HB1), (HB2)*, and let $x, y \in \bar{\Omega}$ and ϕ_a be defined in Definition* A.2*. Then*

$$\big|D_x\phi_a(x,y) + D_y\phi_a(x,y)\big| \leqslant K\left(\frac{|x-y|^2}{\varepsilon^2} + \varepsilon^2\right) + 2B, \tag{A.12}$$

$$\big|D_x\phi_a(x,y)\big| + \big|D_y\phi_a(x,y)\big| \leqslant K\left(\varepsilon^2 + \frac{|x-y|^2}{\varepsilon^2} + \frac{|x-y|}{\varepsilon^2}\right) + 2B. \tag{A.13}$$

Furthermore if $\frac{1}{\varepsilon^2}|x-y|^2 \leqslant C$ and A big enough (independent of a, ε, B) then

$$\big|D_x\phi_a(x,y)\big|, \big|D_y\phi_a(x,y)\big| \geqslant \frac{|x-y|}{2\varepsilon^2}\big(1 - \varepsilon^2\big[B + K(1+A)\big]\big) - K\big(\varepsilon^2 + B\big), \tag{A.14}$$

and if in addition $\varepsilon^2 \leqslant [2B + K(1+A)]^{-1}$ then

$$\big|D_x\phi_a(x,y) + D_y\phi_a(x,y)\big| \leqslant K\big|D_x\phi_a(x,y)\big| \wedge \big|D_y\phi_a(x,y)\big||x-y| + K\big(\varepsilon^2 + B\big). \tag{A.15}$$

Finally, if $|x-y| \leqslant K\eta\varepsilon$ and $0 < \eta, \varepsilon, a \leqslant 1$ then

$$\begin{aligned} &D^2\phi_a(x,y) \\ &\quad \leqslant \frac{K}{\varepsilon^2}\left(1 + \frac{\varepsilon}{a}\eta^3\right)\begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{pmatrix} + K\left(\left(1+\frac{\varepsilon}{a}\eta^3\right)\big(\eta^2+\varepsilon^2\big) + B\right)\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}. \end{aligned} \tag{A.16}$$

Proof. To simplify the computations, we make a change of variables and define $\Phi_{a,\varepsilon}$ by

$$\phi_a(x,y) = \Phi_{a,\varepsilon}(X,Y,Z,T),$$

where $X = \frac{x+y}{2}$, $Y = x - y$, $Z = d(x) - d(y)$, $T = d(x) + d(y)$. We see that

$$\Phi_{a,\varepsilon}(X,Y,Z,T) = \frac{|Y|^2}{\varepsilon^2} - C_{2,a}\left(X, \frac{2Y}{\varepsilon^2}\right)Z + \frac{A}{\varepsilon^2}Z^2 - BT, \tag{A.17}$$

and straightforward computations lead to

$$\begin{aligned}
D_X\Phi_{a,\varepsilon}(X,Y,Z,T) &= -D_x C_{2,a}\left(X, \frac{2Y}{\varepsilon^2}\right)Z, \\
D_Y\Phi_{a,\varepsilon}(X,Y,Z,T) &= \frac{2Y}{\varepsilon^2} - \frac{2}{\varepsilon^2} D_p C_{2,a}\left(X, \frac{2Y}{\varepsilon^2}\right)Z, \\
D_Z\Phi_{a,\varepsilon}(X,Y,Z,T) &= -C_{2,a}\left(X, \frac{2Y}{\varepsilon^2}\right) + \frac{2A}{\varepsilon^2}Z, \\
D_T\Phi_{a,\varepsilon}(X,Y,Z,T) &= -B,
\end{aligned} \tag{A.18}$$

and

$$D_x\phi_a(x,y) = \frac{1}{2}D_X\Phi_{a,\varepsilon} + D_Y\Phi_{a,\varepsilon} + D_Z\Phi_{a,\varepsilon}Dd(x) + D_T\Phi_{a,\varepsilon}Dd(x), \tag{A.19}$$

$$D_y\phi_a(x,y) = \frac{1}{2}D_X\Phi_{a,\varepsilon} - D_Y\Phi_{a,\varepsilon} + D_Z\Phi_{a,\varepsilon}\big(-Dd(y)\big) + D_T\Phi_{a,\varepsilon}Dd(y). \tag{A.20}$$

First note that estimate (A.15) easily follows from (A.12) and (A.14). We start by proving estimate (A.12). Using the fact that $n = -Dd$, we see that

$$D_x\phi_a(x,y) + D_y\phi_a(x,y) = D_X\Phi_{a,\varepsilon} + D_Z\Phi_{a,\varepsilon}\big(n(y) - n(x)\big) + B\big(n(x) + n(y)\big).$$

The first term on the right-hand side can be estimated by (A.7) of Lemma A.2,

$$|D_X\Phi_{a,\varepsilon}| \leqslant K\left(1 + \frac{|Y|}{\varepsilon^2}\right)\big|d(x) - d(y)\big|,$$

and the second term by (A.6),

$$\left|C_{2,a}\left(X, \frac{2Y}{\varepsilon^2}\right)\right| \leqslant K\left(1 + \frac{|Y|}{\varepsilon^2}\right) \quad \text{and} \quad |D_Z\Phi_{a,\varepsilon}| \leqslant K\left(1 + \frac{|Y|}{\varepsilon^2}\right) + \frac{2A}{\varepsilon^2}\big|d(x) - d(y)\big|.$$

Combining these estimates, using the regularity of d, and $Y = x - y$ then leads to

$$\big|D_x\phi_a(x,y) + D_y\phi_a(x,y)\big| \leqslant \big(K + 2A|Dd|_0\big)\left(1 + \frac{|Y|}{\varepsilon^2}\right)|x - y| + 2|Dd|_0 B,$$

and estimate (A.12) follows from Young's inequality. In a similar way, we can also prove (A.13).

We proceed to prove the lower bound (A.14) and start by estimating $|D_x\phi_a(x,y) - D_y\phi_a(x,y)|$. Observe that

$$\begin{aligned} D_x\phi_a(x,y)-D_y\phi_a(x,y) &= 2\left[\frac{2(x-y)}{\varepsilon^2}-\frac{2}{\varepsilon^2}D_pC_{2,a}\left(X,\frac{2Y}{\varepsilon^2}\right)\big(d(x)-d(y)\big)\right] \\ &\quad + \big(Dd(x)+Dd(y)\big)\left[-C_{2,a}\left(X,\frac{2Y}{\varepsilon^2}\right)+\frac{2A}{\varepsilon^2}\big(d(x)-d(y)\big)\right] \\ &\quad - B\big(Dd(x)-Dd(y)\big). \end{aligned}$$

Now using a Taylor expansion and regularity of d, we see that

$$\big(Dd(x)+Dd(y)\big)\cdot(x-y)\leqslant 2\big(d(x)-d(y)\big)+\frac{1}{8}\big|D^3d\big|_0|x-y|^3,$$

and after applying also (A.6) we get

$$\begin{aligned} &\big(D_x\phi_a(x,y)-D_y\phi_a(x,y)\big)\cdot(x-y) \\ &\quad\geqslant 4\frac{|x-y|^2}{\varepsilon^2}-\frac{4}{\varepsilon^2}D_pC_{2,a}\left(X,\frac{2Y}{\varepsilon^2}\right)\cdot(x-y)\big(d(x)-d(y)\big) \\ &\qquad + 2C_{2,a}\left(X,\frac{2Y}{\varepsilon^2}\right)\big(d(x)-d(y)\big)+\frac{4A}{\varepsilon^2}\big(d(x)-d(y)\big)^2 \\ &\qquad - \big|D^3d\big|_0|x-y|^3\left(\frac{A}{\varepsilon^2}|Dd|_0|x-y|+K\left(1+\frac{|x-y|}{\varepsilon^2}\right)\right) \\ &\qquad - B\big(Dd(x)-Dd(y)\big)\cdot(x-y). \end{aligned}$$

Using Young's inequality as in Lemma A.3 and taking A even bigger if necessary (but not depending on a,ε,B), we have ($\varepsilon\leqslant 1$)

$$\begin{aligned} &\big(D_x\phi_a(x,y)-D_y\phi_a(x,y)\big)\cdot(x-y) \\ &\quad\geqslant \frac{|x-y|^2}{\varepsilon^2}-B|Dd|_0|x-y|^2-K(1+A)\big(1+\varepsilon^2\big)\frac{|x-y|^4}{\varepsilon^2} \\ &\quad = \frac{|x-y|^2}{\varepsilon^2}\left(1-\varepsilon^2\left(B+K(1+A)\frac{|x-y|^2}{\varepsilon^2}\right)\right), \end{aligned}$$

and Cauchy–Schwarz inequality immediately yields

$$\big|D_x\phi_a(x,y)-D_y\phi_a(x,y)\big|\geqslant\frac{|x-y|}{\varepsilon^2}\left(1-\varepsilon^2\left(B+K(1+A)\frac{|x-y|^2}{\varepsilon^2}\right)\right).$$

Now (A.14) follows by combining the last inequality and (A.12),

$$\big|D_x\phi_a(x,y)\big|\geqslant\frac{|x-y|}{2\varepsilon^2}\left(1-\varepsilon^2\left(B+K+K(1+A)\frac{|x-y|^2}{\varepsilon^2}\right)\right)-K\big(\varepsilon^2+B\big).$$

Now we prove estimate (A.16). A straightforward calculation using (A.18) yields

$$D_{XX}\Phi_{a,\varepsilon}(X,Y,Z,T) = -D^2_{xx}C_{2,a}\left(X,\frac{2Y}{\varepsilon^2}\right)Z,$$

$$D_{XZ}\Phi_{a,\varepsilon}(X,Y,Z,T) = -D_x C_{2,a}\left(X,\frac{2Y}{\varepsilon^2}\right),$$

$$D_{XY}\Phi_{a,\varepsilon}(X,Y,Z,T) = -\frac{2}{\varepsilon^2}D_{xp}C_{2,a}\left(X,\frac{2Y}{\varepsilon^2}\right)Z,$$

$$D_{YY}\Phi_{a,\varepsilon}(X,Y,Z,T) = \frac{2}{\varepsilon^2}\,\mathrm{Id} - \left(\frac{2}{\varepsilon^2}\right)^2 D_{pp}C_{2,a}\left(X,\frac{2Y}{\varepsilon^2}\right)Z,$$

$$D_{YZ}\Phi_{a,\varepsilon}(X,Y,Z,T) = -\frac{2}{\varepsilon^2}D_p C_{2,a}\left(X,\frac{2Y}{\varepsilon^2}\right),$$

$$D_{ZZ}\Phi_{a,\varepsilon}(X,Y,Z,T) = \frac{2A}{\varepsilon^2}\,\mathrm{Id},$$

$$D_{TX}\Phi_{a,\varepsilon} = D_{TY}\Phi_{a,\varepsilon} = D_{TZ}\Phi_{a,\varepsilon} = D_{TT}\Phi_{a,\varepsilon} = 0.$$

We will estimate these terms using Lemma A.2 and (A.1). For example, one has

$$\left|D_{XX}\Phi_{a,\varepsilon}(X,Y,Z,T)\right| \leqslant K\frac{\Gamma^2}{\Lambda}\left|d(x)-d(y)\right| \leqslant K\frac{\Gamma^2}{\Lambda}\left(\varepsilon^2\left|p\cdot n(X)\right| + \left|D^3 d\right|_0 |x-y|^3\right),$$

where $p = \frac{2Y}{\varepsilon^2}$. In our case $\Lambda = [a^2 + (p\cdot n(X))^2]^{\frac{1}{2}}$, and $\Gamma = (1+|p|^2)^{\frac{1}{2}}$, and hence

$$\begin{aligned}\left|D_{XX}\Phi_{a,\varepsilon}(X,Y,Z,T)\right| &\leqslant K\left(\varepsilon^2\Gamma^2 + \frac{\Gamma^2}{a}|x-y|^3\right)\\ &\leqslant K\left(\varepsilon^2 + \frac{|x-y|^2}{\varepsilon^2}\right)\left(1+\frac{\varepsilon}{a}\frac{|x-y|^3}{\varepsilon^3}\right).\end{aligned} \tag{A.21}$$

By carefully doing computations like above, one can prove that

$$\left|D_{YY}\Phi_{a,\varepsilon}(X,Y,Z,T)\right| \leqslant \frac{K}{\varepsilon^2}\left(1+\frac{\varepsilon}{a}\frac{|x-y|^3}{\varepsilon^3}\right), \tag{A.22}$$

$$\left|D_{YZ}\Phi_{a,\varepsilon}(X,Y,Z,T)\right|, \left|D_{ZZ}\Phi_{a,\varepsilon}(X,Y,Z,T)\right| \leqslant \frac{K}{\varepsilon^2}, \tag{A.23}$$

$$\left|D_{XZ}\Phi_{a,\varepsilon}(X,Y,Z,T)\right| \leqslant K\left(1+\frac{|x-y|}{\varepsilon^2}\right), \tag{A.24}$$

$$\left|D_{XY}\Phi_{a,\varepsilon}(X,Y,Z,T)\right| \leqslant K\left(1+\frac{|x-y|}{\varepsilon^2}\right)\left(1+\frac{\varepsilon}{a}\frac{|x-y|^3}{\varepsilon^3}\right). \tag{A.25}$$

Now we compute the matrix $D^2\phi_a(x,y)$ from (A.19) and (A.20):

$$D^2\phi_a(x,y) = M_1 + M_2 + M_3 + M4 + M_5 + M_6 + M_7 + M_8,$$

where

$$M_1 = \begin{pmatrix} D^2_{YY}\Phi_{a,\varepsilon} & -D^2_{YY}\Phi_{a,\varepsilon} \\ -D^2_{YY}\Phi_{a,\varepsilon} & D^2_{YY}\Phi_{a,\varepsilon} \end{pmatrix},$$
$$M_2 = \frac{1}{4}\begin{pmatrix} D^2_{XX}\Phi_{a,\varepsilon} & D^2_{XX}\Phi_{a,\varepsilon} \\ D^2_{XX}\Phi_{a,\varepsilon} & D^2_{XX}\Phi_{a,\varepsilon} \end{pmatrix},$$
$$M_3 = \begin{pmatrix} D^2_{XY}\Phi_{a,\varepsilon} & 0 \\ 0 & -D^2_{XY}\Phi_{a,\varepsilon} \end{pmatrix},$$
$$M_4 = D^2_{ZZ}\Phi_{a,\varepsilon}\begin{pmatrix} Dd(x)\otimes Dd(x) & -Dd(x)\otimes Dd(y) \\ -Dd(y)\otimes Dd(x) & Dd(y)\otimes Dd(y) \end{pmatrix},$$
$$M_5 = D^2_{ZY}\Phi_{a,\varepsilon}\otimes\begin{pmatrix} 2Dd(x) & -Dd(x)-Dd(y) \\ -Dd(x)-Dd(y) & 2Dd(y) \end{pmatrix},$$
$$M_6 = D^2_{ZX}\Phi_{a,\varepsilon}\otimes\begin{pmatrix} Dd(x) & \frac{1}{2}(Dd(x)-Dd(y)) \\ \frac{1}{2}(Dd(x)-Dd(y)) & -Dd(y) \end{pmatrix},$$
$$M_7 = D_Z\Phi_{a,\varepsilon}\begin{pmatrix} D^2d(x) & 0 \\ 0 & -D^2d(y) \end{pmatrix},$$
$$M_8 = -B\begin{pmatrix} D^2d(x) & 0 \\ 0 & D^2d(y) \end{pmatrix}.$$

It can easily be seen that M_1 (use (A.22)), M_2 (use (A.21)), and M_8 can be bounded from above by a matrix of the form (A.16). Note that

$$\big(M_3(\zeta,\kappa),(\zeta,\kappa)\big) = \big(D^2_{XY}\Phi_{a,\varepsilon}(\zeta-\kappa),(\zeta+\kappa)\big) \leqslant \frac{1}{\theta^2}\big|D^2_{XY}\Phi_{a,\varepsilon}\big|^2|\zeta-\kappa|^2 + \theta^2|\zeta+\kappa|^2,$$

where $\theta = \eta\sqrt{1+\frac{\varepsilon}{a}\eta^3}$ and hence by (A.24) M_3 is also bounded from above by (A.16). Now we write

$$M_7 = D_Z\Phi_{a,\varepsilon}\left(\begin{pmatrix} D^2d(x) & 0 \\ 0 & -D^2d(x) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & D^2d(x)-D^2d(y) \end{pmatrix}\right),$$

and handle the first part of M_7 like we did with M_3. The second part can handled using the $W^{3,\infty}$-regularity of d together with the first-order estimates of $\Phi_{a,\varepsilon}$. We proceed with M_4:

$$\begin{aligned}\big(M_4(\zeta,\kappa),(\zeta,\kappa)\big) &= \frac{2A}{\varepsilon^2}\big(\zeta\cdot Dd(x)-\kappa\cdot Dd(y)\big)^2 \\ &\leqslant \frac{2A}{\varepsilon^2}|Dd|_0^2|\zeta-\kappa|^2 + \big|D^2d\big|_0^2\eta^2\big(|\zeta|^2+|\kappa|^2\big).\end{aligned}$$

The two remaining terms can be treated analogously using also (A.22) and (A.24). This ends the proof of Lemma A.5. □

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